A Use of Quantum Numbers With Beat Frequencies in Psychoacoustics.

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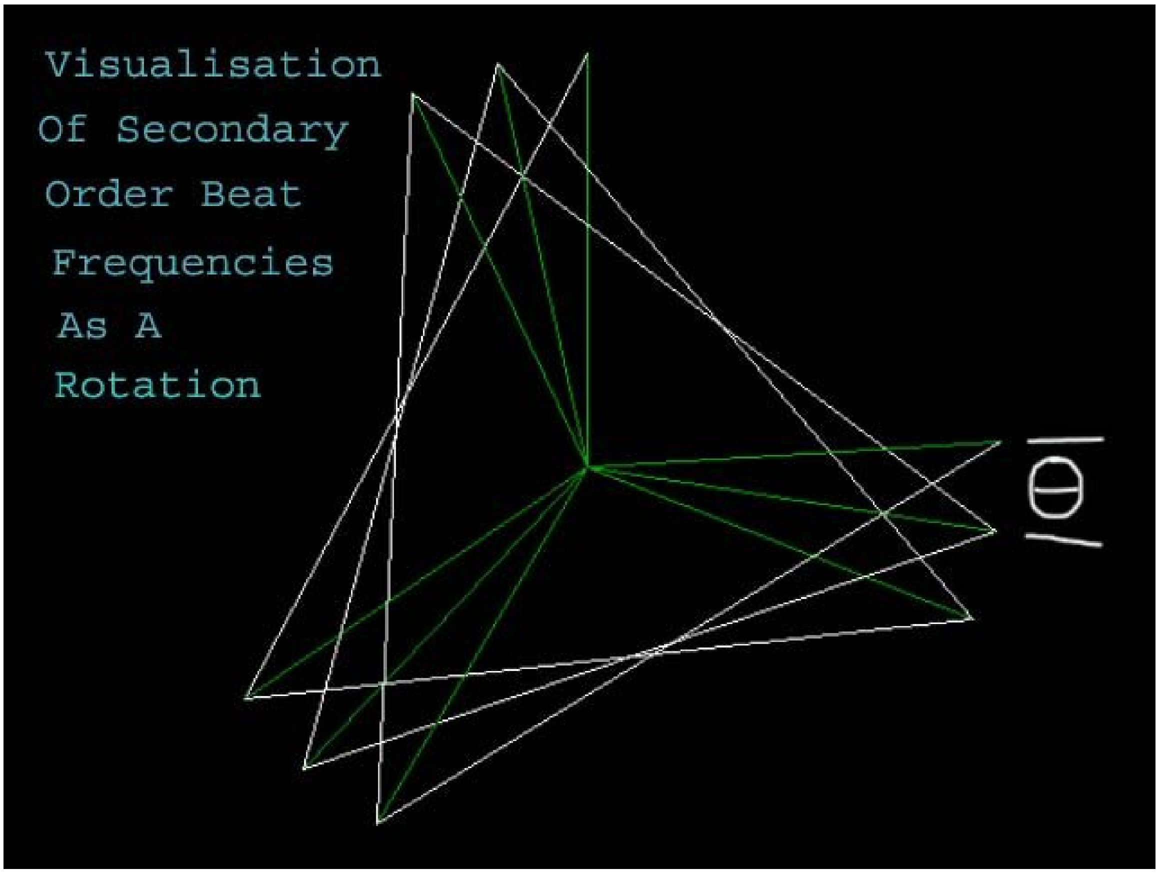
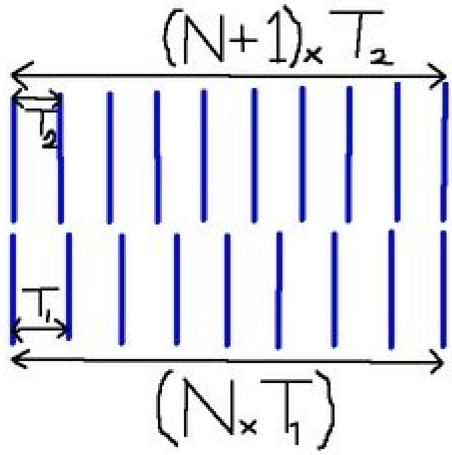


We have searched this topic and only found theory of primary order 'beating' frequencies which = difference between two component frequencies played.
Apologies in advance if this subject is already known to you, for it does seem obvious, a surprisingly simple expression but it doesn't seem to be written about much.
This subject deals with the secondary order low frequency oscillations heard with two frequencies sounded together in a integer ratio but slightly off, such as $(3+\delta)/2$, $(5+\delta)/3$ etc...

Considering the visual model of 2 orbiting particles, the conjunction frequency, when the particles align radially = subtraction between the two frequencies.

Here is a basic proof:



T1 = timeperiod of first planet in a coupled orbital system
T2 = timeperiod of second planet in the same system
N = number of orbits it takes for an alignment (conjunction to occur)

Since sound is matter in vibration, and mutually, matter in vibration is sound, then the orbital motion of planets are really a very low frequency form of sound.
By definition, timeperiod = 1 / frequency so in the diagram in a simple representation of planetary orbits, the lower series of lines associated with T2 (Timeperiod of planet 2) has one more orbit to happen before both planets align.
So $(N+1)*(T2) = N*(T1)$
To find N, : $(N+1)/N = (T1)/(T2)$
So $1 + (1/N) = (T1)/(T2)$
$(1/N) = (T1-T2)/T2$
ie: $N = T2/(T1-T2)$
So we know an expression for N.
We know that in the diagram N*T1 is the timeperiod of conjunction, when the planets align, so letting this = Tc:
$Tc = N*T1 = (T1*T2)/(T1-T2)$
So since frequency = 1/timeperiod by definition, we can rearrange equation to give:
$1/Tc = (T1-T2)/(T1*T2)$
Which is $1/Tc = (1/T2)-(1/T1)$

So frequency $Fc = F2 - F1$

So we now know conjunction frequency, Fc = difference between two planet frequencies F1 & F2

So as an example Considering two component frequencies being close to a numerical integer ratio like 8/5

Letting bottom frequency, $fa = 5$ and top frequency $fb = 8$, and then increment $fb = 8$ to $(8+\delta)$ where δ is a small change
The primary difference frequency (or conjunction alignment frequency) = $(8+\delta)-5 = (3+\delta)$

As shown in the picture before,
the primary angle swept out = $5/(3+\delta)$ which is $fa/(fb-fa+\delta)$
So how many conjunctions will it take to have a conjunction point rotated round a number of full turns to be back close to the starting point?
Well to get the conjunction angle=$5/(3+\delta)$ back to near the start angle, can multiply by 3 to give close to 5 complete full rotations.

To find the angle of displacement θ as shown in the picture:
$\theta = 5-[3*(5/(3+\delta))]$
or
$\theta = fa-[(fb-fa)*(fa/(fb-fa+\delta))]$

Simplifying equation gives :

$\theta = fa*\delta/(fb+\delta-fa)$

The secondary order time period, $T\beta$ = (primary order time period)/ θ

SO if primary order time period = $T\alpha = (1/fo)*[fa/((fb+\delta)-fa)]$ {because timeperiod = 1/frequency}{f0 = fundamental base frequency in relation to frequency component , fa}

Then expression for $T\beta = (1/fo)*[fa/(fb+\delta-fa)]*[(fb+\delta-fa)/(fa*\delta)]$
Terms cancel giving :

$T\beta = 1/(\delta*fo)$

By definition, frequency = 1/timeperiod
So for secondary order beat frequency,$F\beta$:

$F\beta = \delta*fo$

This can be observed as an additional periodicity in the waveform.

We have experimented with a few ratios like 3/2 , 5/3, 5/2, 8/5 and this equation seems consistent with results.

And if we want to introduce a small displacement, ζ, from fb/fa, then we can equate $(fb+\delta)/fa = (fb/fa)+\zeta$

Which can be simplified: $\zeta * fa = \delta$

So secondary order timeperiod, $T\beta = 1/(fo*fa*\zeta)$

and secondary order frequency,

$$F\beta = fo*fa*\zeta$$

As has been said , the experimental results using wave editing software to observe the low frequency periodicity visually have been very very accurate so far.

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Basic Primary Order Beat Frequencies:

Someone talked about "Basic" beat ratios a while back, and to some we are probably showing a result which you already know.
Letting a Triad of sounded intervals be given frequency values of : I, J, K, where these values can be represented as integer fractions.
Let Basic Beat frequencies be assigned as values A, B, C
$C = K-J$
$B = J-I$
$A = K-I$
We want to input values for the beat frequencies to produce results for the sounded frequencies I, J, K
So if we let beat frequencies:
$C = v/u$
$B = s/r$
$A = q/p$
And letting one of the intervals played in the triad, $I = 1$
then resultant played frequencies are as integer ratios:
$K = (rv+(s+r)u)/(ru) +\Omega$
$J = (s+r)/r +\Omega$
$I = 1 + \Omega$
alternatively expressed as:-
$K = (q+p)/p + \Omega$
$J = (u(q+p)-pv)/(pu) + \Omega$
$I = 1 +\Omega$

And letting one of the intervals played in the triad, $J = 1$
then resultant played frequencies are as integer ratios:
$K = (v+u)/u +\Omega$
$J = 1 +\Omega$
$I = (r-s)/r +\Omega$ tick
alternatively expressed as:-
$K = (r(q+p)+ps)/pr + \Omega$
$J = 1 + \Omega$
$I = (pv +u(p-q))/pu +\Omega$

And letting one of the intervals played in the triad, $K = 1$
then resultant played frequencies are as integer ratios:
$K = 1 +\Omega$
$J = (u-v)/u +\Omega$
$I = (u(r-s)-rv)/(ur) +\Omega$
alternatively expressed as:-
$K = 1 +\Omega$
$J = (ps +r(q-p))/pr +\Omega$
$I = (p-q)/p +\Omega$ (equations have been corrected due to error)

The arbitrary parameter Ω seems right to add since for example:
$I = 3, J = 4, K = 5$ produces beats $(5-3) = 2, (5-4) = 1, (4-3) = 1$
$+\Omega$ could make $I = 17, J = 18, K = 19$ and an identical Format of Beats.

So here is a first example letting $I = 1$, with this following line solved in this particular example:
$s=4, r=5, v=5, u=6, J=(s+r)/r, K = (rv+(s+r)u)/(ru)$
Beat frequencies:
$C =v/u= 5/6$
$B =s/r= 4/5$
$A = K - I = (rv+su)/ru = 49/30$

Tangible Intervals in triad sounded are
$K = 79/30 +\Omega$
$J = 9/5 +\Omega$
$I = 1 +\Omega$

Alternatively by swapping values:
$s=5, r=6, v=4, u=5, J=(s+r)/r, K=((s+r)*u+r*v)/(r*u),$
$C = 4/5$
$B = 5/6$
$A = K - I = u(r + s)/rv = 54/25$

Tangible Intervals in triad sounded are
$K = 79/30 +\Omega$
$J = 11/6 +\Omega$
$I = 1 +\Omega$

Now we may wish to make a value of Ω such that one note of the played triad is in some ratio with a selected Basic Beat frequency so $J/B = n/m$
We use previous equations,
$B = s/r$
$J = (s+r)/r +\Omega$
Obviously one note in the triad played= $J = (n/m)*(s/r)$
But using a previous equation for another note in the played triad
$K=(r*v+(s+r)*u)/(r*u)+\Omega$
where other Basic beat frequency, $C = v/u$
we find other note = $K = (mrv + nsu)/(mru)$
And other note in triad = $I = s(n-m)/(mr)$

So as an example if we let basic beat frequencies
$C = v/u =4/1$
$B = s/r = 3/1$

Value of ratio J/B=n/m=	2/1	5/2	3/1	7/1	8/1	9/1	12/1
notes in Triad played:							
K = (3n+4m)/m=	10	23/2	13	25	28	31	40
J = 3n/m=	6	15/2	9	21	24	27	36
I = (3n - 3m)/m	3	9/2	6	18	21	24	33

But this is one of many examples yet to be explored...

Instead, we can make a ratio between the Same 'J' note in the triad sounded with the other beat frequency, 'C'
Now instead Letting J/C = n/m
We use previous equations,
C = v/u
J = (s+r)/r +Ω
Of course one note in the triad played= J = (n/m)*(v/u)
But using the same previous equation for another note in the played triad
K=(r*v+(s+r)*u)/(r*u)+Ω
where other Basic beat frequency, B = s/r
we find other note = K = v(n+m)/(mu)
And other note in triad = I = (nrv - msu)/(mru)

So as an example if we let basic beat frequencies
C = v/u =5/1
B = s/r = 4/1

Value of ratio J/B=n/m=	2/1	3/1	4/1	5/1	6/1	7/1	8/1
notes in Triad played:							
K = v(n+m)/(mu)	15	20	25	30	35	40	45
J = (n/m)*(v/u)=	10	15	20	25	30	35	40
I = (nrv - msu)/(mru)=	6	11	16	21	26	31	36

And, we can make a ratio between the Same 'J' note in the triad sounded with the other beat frequency, 'A'
Now instead Letting $J/A = n/m$
We use previous equations,
$A = q/p$
$J = (u(q+p)-pv)/(pu) + \Omega$

Of course one note in the triad played= $J = (n/m)*(q/p)$
But using the same previous equation for another note in the played triad
$K = (q+p)/p + \Omega$

where other Basic beat frequency, $C = v/u$
we find other note = $K = (mpv + nqu)/(mpu)$
And other note in triad = $I = (mpv +qu(n-m))/mpu$

Now for the Next Format of 3

So Also we may wish to make a value of Ω such that a different note of the played triad is in some ratio with a selected Basic Beat frequency so $K/B = n/m$
We use previous equations,
$B = s/r$
$K = (rv+(s+r)u)/(ru) + \Omega$
Obviously one note in the triad played= $K = (n/m)*(s/r)$
But using a previous equation for another note in the played triad
$J = (s+r)/r + \Omega$
where other Basic beat frequency, $C = v/u$
we find other note = $J = (nsu – mrv)/(mru)$
And other note in triad = $I = (us(n-m) – mrv)/(mru)$

Instead, we can make a ratio between the Same 'K' note in the triad sounded with the other beat frequency, 'C'
Now instead Letting $K/C = n/m$
We use previous equations,
$C = v/u$
$K=(r*v+(s+r)*u)/(r*u)+\Omega$
Of course one note in the triad played= $K = (n/m)*(v/u)$
But using the same previous equation for another note in the played triad
$J = (s+r)/r + \Omega$ where other Basic beat frequency, $B = s/r$
we find other note = $J = v(n-m)/mu$
And other note in triad = $I = (rv(n-m) – msu)/mru$

And, we can make a ratio between the Same 'K' note in the triad sounded with the other beat frequency, 'A'
Now instead Letting $K/A = n/m$
We use previous equations,
$A = q/p$
$K = (q+p)/p + \Omega$
Of course one note in the triad played= $K = (n/m)*(q/p)$
But using the same previous equation for another note in the played triad
$J = (u(q+p)-pv)/(pu) + \Omega$

where other Basic beat frequency, $C = v/u$
we find other note = $J = (nqu – mpv)/(mpu)$
And other note in triad = $I = q(n-m)/pm$

And The Last 3 Format.

And we may wish to make a value of Ω such that the final note of the 3 of the played triad is in some ratio with a selected Basic Beat frequency so $I/B = n/m$
We use previous equations,
$B = s/r$
$I = 1 + \Omega$
Obviously one note in the triad played= $I = (n/m)*(s/r)$
But using previous equations for other notes in the played triad
$J = (s+r)/r + \Omega$
$K = (rv+(s+r)u)/(ru) + \Omega$
where other Basic beat frequency, $C = v/u$
we find other note = $J = s(n + m)/mr$
And other note in triad = $K = (s(nu + mu) + mrv)/mru$

Instead, we can make a ratio between the Same 'I' note in the triad sounded with the other beat frequency, 'C' , Now instead Letting $I/C = n/m$, We use previous equations,
$C = v/u$
$I = 1 + \Omega$
Of course one note in the triad played= $I = (n/m)*(v/u)$
But using the same previous equation for another note in the played triad
$J = (s+r)/r + \Omega$
$K = (rv+(s+r)u)/(ru) + \Omega$
where other Basic beat frequency, $B = s/r$
we find other note = $J = (nrv + msu)/mru$
And other note in triad = $K = (rv(n+m) + msu)/mru$

And, we can make a ratio between the Same 'I' note in the triad sounded with the other beat frequency, 'A' ,Now instead Letting $I/A = n/m$,We use previous equations,
$A = q/p$
$I = 1 + \Omega$
Of course one note in the triad played= $I = (n/m)*(q/p)$
But using the same previous equation for another note in the played triad
$J = (u(q+p)-pv)/(pu) + \Omega$
$K = (q+p)/p + \Omega$
where other Basic beat frequency, $C = v/u$
we find other note = $J = (qu(n+m) - mpv)/mpu$
And other note in triad = $K = q(n+m)/mp$

Now that we have dealt a little with Basic Primary Order beats, we will move onto Secondary order Beats which can be thought of as an additional periodicity in the structure of the waveform.

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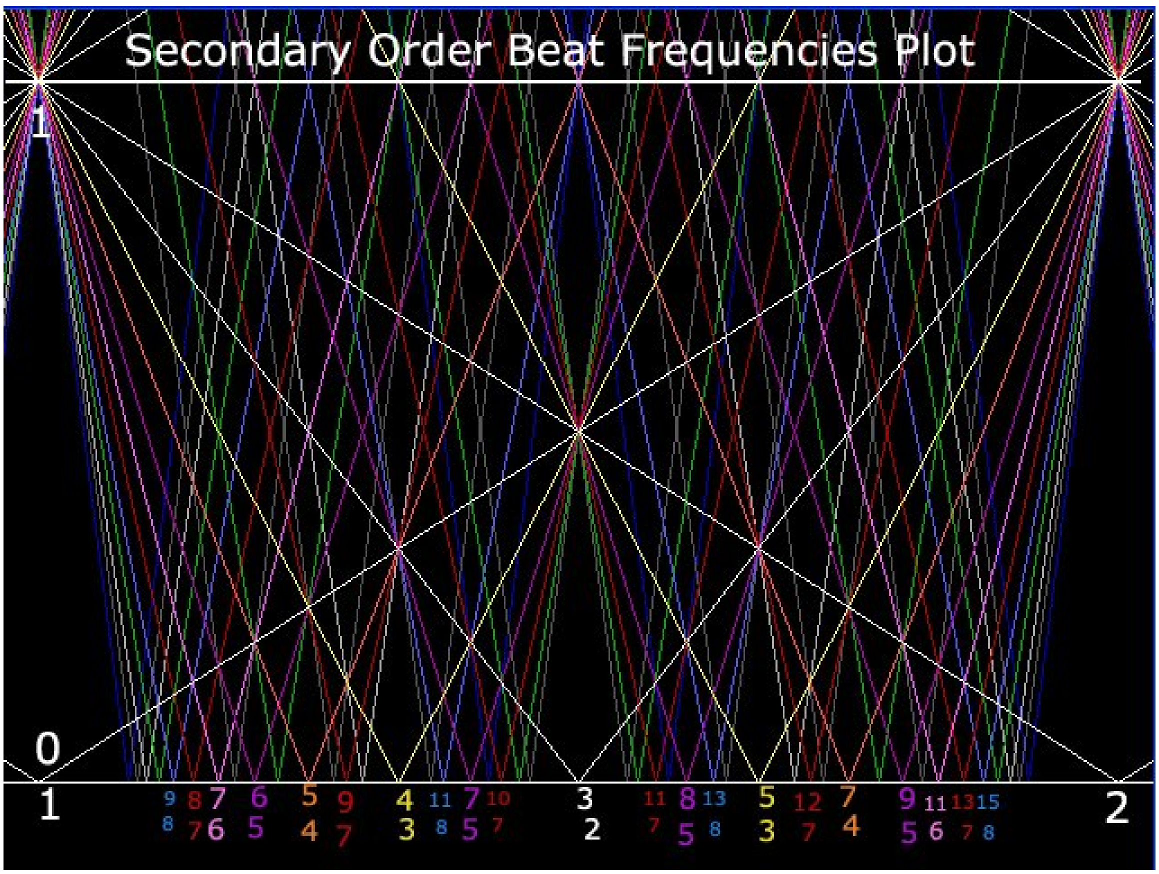
<u>Secondary Order Beat Frequencies</u>

Here is a diagram of some particular ratios for some intervals in the psychoacoustic map and the resultant secondary order beating frequency for integer ratio = b/a displaced by variable, ζ so fundamental frequency= fo is sounded & {fo*[(b/a)+ ζ]} is sounded simultaneously.

Resultant secondary order beat frequency Fβ =modulas[ζ*a] is plotted for vertical axis in the region roughly from 0 to 1. The horizontal axis is for (b/a)+ ζ in approximately the region from 1 to 2

An infinite number of ratios should have been included at disappearing intensities which will be made apparent in future collaborative work.



There are some great uses for this if this hypothesis is indeed correct. Experimental evidence hasn't showed any flaws as yet.
The secondary order beat frequency could be in a harmonious relationship with the lower frequency or the higher in the interval.
Take for example a mistuned perfect fifth in music = $(3/2)+ \zeta = (fb/fa)+ \zeta$
Letting Secondary order beat frequency $F\beta = (m/n)$ a pair of integers

Starting again: integers b/a is interval
mistuned interval, $I = (b/a)+\zeta$

secondary order beat frequency
$F\beta = fo*\zeta*a$ where fo is fundamental frequency

Letting $F\beta = fo*(m/n)$ where n & m are a pair of integers

So:
$(m/n) = \zeta*a$

$\zeta= m/(n*a)$

To find complete interval $I= (b/a)+\zeta$
$I = (b/a)+(m/(n*a))$
$I = ((a*b*n) + (a*m))/(n*a^2)$

so
for interval expressed as integer fraction:

Interval, $I+ = ((b*n)+m)/(a*n)$ When $\zeta+$

Interval, $I- = ((b*n)-m)/(a*n)$ When $\zeta-$

So letting secondary order beat frequency $F\beta/fo = (m/n)$ be a much lower ratio in relationship with the (3/2) interval
Say for example for close integer interval, $3/2 = (b/a)$ Letting the 3 brought down quite a few octaves to = 3/64
so m = 3 & n = 64,
then
mistuned interval , $I =((3*64)+3)/(2*64)$

$I = 195/128 = 728.796$cents
For secondary order beat frequency $F\beta = fo*3/64$

Primary order $F\alpha =(I-1) =(b/a)+\zeta -1$
$= ((b*n)+m-(a*n))/a*n =$

$F\alpha = f0*(n(b-a) +m)/ a*n$ [expressed as integer ratio]

in this example $F\alpha = fo*67/128$

Alternatively, We can also find another expression when $F\alpha$, primary order beat frequency divided by $F\beta$, secondary order beat frequency = whole number =R
So

$F\alpha = R*F\beta$

from our previous equations then :

$(b/a)+\zeta-1 = R*a*\zeta$
This is when mistuned interval , $I > (b/a)$ where as designated before (b/a) is reduced integer fraction thought associated with and close to mistuned interval where $I = (b/a)+ \zeta$

As an integer fraction:
$\zeta = (b-a)/(a*((R*a)-1))$

if we let R = integer fraction = (g/h)
then:

$\zeta+ = h*(b-a) / (a*((a*g)-h))$ expressed as integer fraction.

This $\zeta+$ is when interval sounded , $I >$ associated appropriate reduced integer fraction (b/a)

When interval, $I < (b/a)$ then there will be a different value for ζ since the ratio of primary order beat frequency with secondary order beat frequency is non-symmetric about (b/a)

So

$(b/a)-\zeta-1 = R*a*\zeta$

As an integer fraction:
$\zeta = (b-a)/(a*((R*a)+1))$

if we let R = integer fraction = (g/h)
then:

$\zeta- = h*(b-a) / (a*((a*g)+h))$ expressed as integer fraction.

This $\zeta-$ is when interval sounded , $I <$ associated appropriate reduced integer fraction (b/a)

So in summary, when we have Primary Order conjunction frequency, $F\alpha$, in an integer ratio, (g/h) with Secondary Order beat frequency, $F\beta$, interval sounded , I , has a sharp form and a flat form.

Sharp Form: Interval, $I = (b/a) + (\zeta+)$

$$I = (h - (b*g))/(h - (a*g))$$

Flat Form: Interval, $I =(b/a) - (\zeta-)$

$$I= (h + (b*g))/(h + (a*g))$$

As an example if we want ratio of upper primary order beat frequency with lower secondary order beat frequency = integer fraction g/h,
Let's say g/h = 25/3

For the case where Top integer for initial pair of frequencies played,[b] = 3

And where the other of the pair, the Base integer,[a] = 2

Then $\zeta+ = h*(b-a) / (a*((a*g)-h))$
So $\zeta+ = h*(b-a) / (a*((a*g)-h)) = 3/94$

interval $I+ = (b/a)+\zeta = 72/47$

value for this interval in cents = 1200*log(72/47)/log(2)

I+ = 738.403cents

$F\alpha = I-1 = 25/47$

$F\beta = a*\zeta = 2*3/94 = 3/47$

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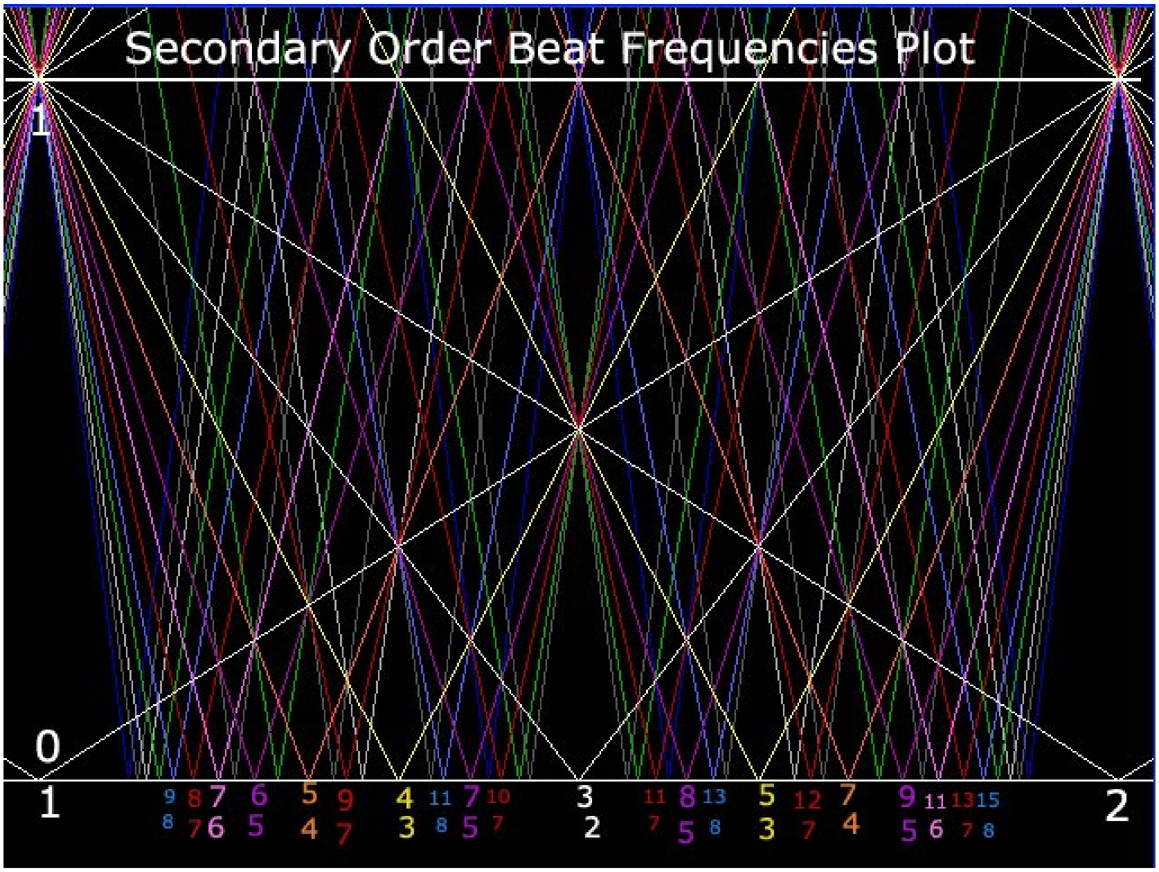


Ratio of A Triad of Secondary Order Beat Frequencies.

Someone Mentioned a while back about beat ratios for a mistuned crushed octave piano , so it got us thinking.
So it is known that for two notes played together, one being = 1, other = $(b/a)+\zeta$ or $(b/a)-\zeta$ {symmetry holds for this}, where b & a are integers & ζ is a value that can thought of as a ratio of absolutely any fraction integers so any continuous value is possible for ζ
 Secondary order beat frequency which is really an additional form of wave structure periodicity = $a*\zeta$
,as shown for particular intervals in the picture. Gradient = bottom integer of interval and they all seem to exist at once but their intensities is still unclear for many theorists I Psychoacoustics.

So choosing 3 component frequency just intonation ratio points that generate a perceived secondary order beat frequency at a single particular chosen interval= I ,
for example $(1/1)+\zeta_1 = I$ & $(3/2)+\zeta_2 = I$ & $(2/1)-\zeta_3 = I$
or $(b/a)+\zeta_1 = I$ & $(h/g)+\zeta_2 = I$ & $(q/p)-\zeta_3 = I$
so:
$\zeta_1 = I-(b/a)$ & $\zeta_2 = I-(h/g)$ & $\zeta_3 =(p/q)-I$

Remembering beat frequencies = $a*\zeta$, $g*\zeta$, $p*\zeta$ we can try to find an interval that exists when:

$(a*\zeta_1)/(f*\zeta_2)= R * (f*\zeta_2)/(p*\zeta_3)$ where R = a ratio thought of as a fraction of some whole numbers
This beat ratio (or same as additional wave periodicity ratio) was selected due to looking at the right-hand region of the 'Secondary Order Beat Frequency Plot' picture shown above.

So substituting into the last equation for values of ζ_1 , ζ_2, ζ_3 shows:
$[a*(I-(b/a))]/(f*[I-(h/g)])= R *[f(I-(h/g))]/(p*[(q/p)-I])$

Rearranging gives a quadratic of some terms for I^2 & I & constant so solving the formula to find interval I:

$I=[\pm\sqrt{ ((4agh-4bg^2)q+(4bgh-4ah^2)p)R+a^2q^2-2abpq+b^2p^2 } +2ghR+aq+bp]/ (2g^2R+2ap)$

This could be thought of as the 'Far Right region ' and there are two results for + or - the square root term

As an example for (b/a) = 1/1, (h/g) = 3/2, (q/p) = 2/1
interval $I = [\{ \pm\sqrt{(4R+1)} \} +12R +3]/ (8R+2)$

Alternatively We could have a known interval, I, and want to evaluate R

$R = ((I * (aq + pb)) - I^2ap - bq) / [(Ig - h)^2]$

And finally the 'Far Left Region'

When the interval becomes less than middle associated beat point (3/2) in this example then ζ_2 becomes negative so:
in this example new equations are $(1/1)+\zeta_1 = I$ & $(3/2)-\zeta_2 = I$ & $(2/1)-\zeta_3 = I$
or $(b/a)+\zeta_1 = I$ & $(h/g)-\zeta_2 = I$ & $(q/p)-\zeta_3 = I$

so $\zeta_1 = I-(b/a)$ & $\zeta_2 = (h/g)-I$ & $\zeta_3=(p/q)-I$

But relations instead are:

$(p*\zeta_3)/(g*\zeta_2)= R *(g*\zeta_2)/(a*\zeta_1)$ where R is some ratio to be selected.

Substituting and solving quadratic for interval, I shows:

Interval, $I =[\pm\sqrt{ ((4agh-4bg^2)q+(4bgh-4ah^2)p)R+a^2q^2-2abpq+b^2p^2 } +2ghR+aq+bp]/(2g^2R+2ap)$

For the example (b/a) = 1/1, (h/g) = 3/2, (q/p) = 2/1

interval I = [{ +-√(4R+1) } +12R +3]/ (8*R+2)

When R = 1 in equilibrium,
interval I = (√5+15)/10 =942.517cents
or mirror about (3/2) interval =2-I+1 = (15-√5)/10=442.487cents

For the middle right region, in the diagram, the line associated with beat point (1/1) is at the top, (2/1) line is in the middle & (3/2) line is at the bottom, so if we want to find a ratio, S
$(a*\zeta_1/(p*\zeta_3)= S *(p*\zeta_3)/(g*\zeta_2)$ Where S is some ratio to be selected.
Interval I = $[(3S-3)\sqrt{(4S+1)}+4S^2 -5S+3] / [(S-2) \sqrt{(4S+1)}+2S^2-5S+2]$

In equilibrium when beat ratio, S = 1 then
Interval I = (1/2)[√5+1] = Phi,the Golden ratio 833.090cents
By the symmetry the other point in the middle left region on the mirror side of the (3/2) point = 2-I+1 in this case
So other interval point = 3-Phi = 560.0665cents
This interval 3-phi is the same ratio between a type of adjacent Fibonacci numbers 1,2,3,5,8,13,21,34,55,89,144 & Lucas numbers,1,3,4,7,11,18,29,47,76,123
The ratio series 29/21,47/34,76/55,123/89….approaches 3-Phi

When beat ratio , S = 1/2
Interval I = (√3+3)/3 =789.004cents
or on the mirror of the 3/2 point, 2-I+1=interval (6-√3)/3=610.297cents

When Beat Ratio, S = 1/3
Interval, I = [3√3√7 + 13]/[2√3√7 +8] = 767.921cents
Or the mirror point, 2-I+1=interval 633.376cents

When Beat ratio, S = 2, interval I = 5/3 or mirror point 2-I+1=4/3

So Now back to the other format beat ratio , R
Alternatively we can know the interval, I, and want to evaluate this so called Triad beat ratio R:

$R = ((I * (aq + pb)) - I^2ap - bq) / [(Ig - h)^2]$

Which is the same for the other far both sides with variables flipped so symmetry holds it seems.

We now may want to define Beat frequencies triad components, A,B,C as integer frequencies, With each one associated with beat frequencies sounded connected with interval point (2/1), (3/2), (1/1) respectively for the far right region as seen in the 'Secondary Order Beat Frequency Plot'
So stacked in order, they form a chord
C highest integer frequency
B middle integer frequency
A lowest integer frequency

So the 3Fold Secondary Order Beat ratio from before = R = (C/B)/(B/A)

$R = (A.C)/B^2$

For the example (b/a) = 1/1, (h/g) = 3/2, (q/p) = 2/1
Now our equation for the interval I is:

$I = [\ B^2(\sqrt{\{ (4AC/B^2)+1 \}}) +12AC +3B^2\ ]/(8AC+2B^2)$

And for the secondary order beat frequencies: since at the start we defined interval, I = $(b/a)+\zeta_1$, and by the proof shown for Secondary Order Beat Frequency, the same as what could be thought of as additional wave structure periodicity = $a\zeta_1$
Beat frequency corresponding at the far left region to lowest beat frequency in triad =
$a\zeta_1 = a.I-b$
We can evaluate the other members of the beats triad for this region similarly
$g\zeta_2 = h-gI$ and for this term we know that for the far right region $g\zeta_2 = gI-h$

$p\zeta_3 = q-pI$

But of course: For some particular interval, I, there can only be one single breed of triad for the Secondary order Beat frequencies relationship in this exemplary case associated with frequency points (b/a)=(1/1), (h/g) = (3/2), (q/p) = (2/1) as shown by the line gradients in the picture ‘Secondary Order Beat Frequency Plot’ above. (unless we are missing something or rather are wary of making a certain statement)

So for fun Let's try Secondary Order beat frequency triad
C = 5
B = 3
A = 2

Simple result, R = 10/9, interval I = 12/7,		or interval I = 9/7,
Secondary order	$a\zeta_1 = 5/7$	$a\zeta_1 = 2/7$
Beating wave periodicity	$g\zeta_2 = 3/7$	$g\zeta_2 = 3/7$
Frequencies	$q\zeta_3 = 2/7$	$q\zeta_3 = 5/7$

The relationship holds, must be a fluke !

Next we'll try Secondary Order beat frequency triad
C = 8
B = 5
A = 3

Simple result,R = 24/25, interval I = 19/11,		or interval I = 14/11,
Secondary order	$a\zeta_1 = 8/11$	$a\zeta_1 = 3/11$
Beating wave periodicity	$g\zeta_2 = 5/11$	$g\zeta_2 = 5/11$
Frequencies	$q\zeta_3 = 3/11$	$q\zeta_3 = 8/11$

Well it looks like fibonnaci number ratios correlate so far !

Now we'll try Secondary Order beat frequency triad
C = 13
B = 8
A = 5

Simple result,R = 65/64, interval I = 31/18,		or interval I = 23/18,
Secondary order	$a\zeta_1 = 13/18$	$a\zeta_1 = 5/18$
Beating wave periodicity	$g\zeta_2 = 8/18$	$g\zeta_2 = 8/18$
Frequencies	$q\zeta_3 = 5/18$	$q\zeta_3 = 13/18$

Again! Lucky choice.

Let's try some more:
C = 3
B = 2
A = 1

Simple result R = 3/4, interval I = 7/4,		or interval I = 5/4
Secondary order	$a\zeta_1 = 3/4$	$a\zeta_1 = 1/4$
Beating wave periodicity	$g\zeta_2 = 2/4$	$g\zeta_2 = 2/4$
Frequencies	$q\zeta_3 = 1/4$	$q\zeta_3 = 3/4$

This works too !

What if for major triad
C= 5
B= 4
A= 3
Then not so simple result, interval $I = (2*(\pm\sqrt{19}) +57)/38$, R = 15/16
Secondary order $a\zeta_1 = (76-8*\sqrt{19})/152$

Beating wave periodicity $g\zeta_2 = 2/(\sqrt{19})$ and +- results for +-squareroot
Frequencies $q\zeta_3 = (19+2*\sqrt{19})/38$
$a\zeta_1 = 0.270584266$….when using $+\sqrt{19}$ term
$g\zeta_2 = 0.458831467$…. when using $+\sqrt{19}$ term
$q\zeta_3 = 0.729415733$… when using $+\sqrt{19}$ term

To test if correct $a\zeta_3 / a\zeta_2 = 1.589724738$ should be either = 5/4 or = 4/3 so lets take $-\sqrt{19}$ instead.
$g\zeta_2 = 0.458831467$…
$q\zeta_3 = 0.270584266$… when using $-\sqrt{19}$ term
Ratios aren't the same for this it seems.
But this looks like the reverse image again but ok, our luck had to run out at some point.

And now for a format of a major triad:
C = 6
B = 5
A = 4

R = 24/25, interval I = 19/11		or interval I = 14/11
Secondary order	$a\zeta_1 = 8/11$	$a\zeta_1 = 3/11$
Beating wave periodicity	$g\zeta_2 = 5/11$	$g\zeta_2 = 5/11$
Frequencies	$q\zeta_3 = 3/11$	$q\zeta_3 = 8/11$

So this didn't match either but we made some good guesses at the start !

So as stated before, we know and can see diagrammatically, that for one particular interval between two notes played together, there exists one and only one beat frequency triad format, if for this case, we use only 3 preferably basic fundamental integer fraction points associated with Strongly Pronounced Secondary Order Beating Frequencies.

So what about just finding a ratio, U of the beat frequencies associated with the (1/1) point and the (3/2) point in the far right region of the 'Secondary Order Beat Frequencies plot'. Using these points because they sound like the most prominent and loudest beats.

$(b/a)+\zeta_1 = I$, interval & $(g/h)+\zeta_2 = I$, interval,

beat ratio $U = a\zeta_1/g\zeta_2=(aI-b)/(h-gI)$
interval $I = (hU+b)/(gU+a)$

so (b/a) is the (1/1) & (g/h) = the (3/2)

interval $I = (2U+1)/(3U+a)$

Using also the (2/1) point for beats $1\zeta 3$

beat Ratio,U	Interval,I	Beats $1\zeta_1$	Beats $2\zeta_2$	Beats $1\zeta_3$
1/5	8/7or13/7	1/7	5/7	6/7
1/4	7/6or11/6	1/6	4/6	5/6
1/3	6/5or9/5	1/5	3/5	4/5
3/8	17/14or26/14	3/14	8/14	11/14
2/5	11/9or16/9	2/9	5/9	7/9
1/2	5/4or7/4	1/4	2/4	3/4
2/3	9/7or12/7	2/7	3/7	5/7
3/4	13/10or17/10	3/10	4/10	7/10
4/5	17/13or22/13	4/13	5/13	9/13
1/1	4/3or5/3	1/3	1/3	2/3
9/8	35/26or43/26	9/26	8/26	17/26
7/6	27/20or33/20	7/20	6/20	13/20
6/5	23/17or28/17	6/17	5/17	11/17
5/4	19/14or23/14	5/14	4/14	9/14
4/3	15/11or18/11	4/11	3/11	7/11
3/2	11/8or13/8	3/8	2/8	5/8
8/5	29/21or34/21	8/21	5/21	13/21
13/8	47/34or55/34	13/34	8/34	21/34
5/3	18/13or21/13	5/13	3/13	8/13
7/4	25/18or29/18	7/18	4/18	11/18
16/9	57/41or66/41	16/41	9/41	25/41
2/1	7/5or8/5	2/5	1/5	3/5
12/5	41/29or46/29	12/29	5/29	17/29
5/2	17/12or19/12	5/12	2/12	7/12
8/3	27/19or30/19	8/19	3/19	11/19
3	10/7or11/7	3/7	1/7	4/7
4	13/9or14/9	4/9	1/9	5/9
5	16/11or17/11	5/11	1/11	6/11
6	19/13or20/13	6/13	1/13	7/13
7	22/15or23/15	7/15	1/15	8/15
8	25/17or26/17	8/17	1/17	9/17

There are two results for the interval, I in this table since there is a symmetry about the point (3/2) in this particular case, So the Beats $1\zeta_1$ swaps with the Beats $1\zeta_3$ and the Beats $2\zeta_2$ stays the same. We can see the pattern as the beat ratio U increases to the next integer. Interval, I, Numerator is increased by 3, & denominator by 2

Here's Some beating integer relations for 13EDO, 16EDO, 17EDO, 19EDO, 22EDO
For two notes played together, the lowest note is called the Base note in this table:

	Base Note	Interval	Beats1*ζ1 (1/1)point	Beats2*ζ2 (3/2)point	Beats1*ζ3 (2/1)point
13EDO:					
Note1:	55	58/55	3	49	52
	73	77/73	4	65	69
Note2:	80	89/80	9	62	71
Note3:	75	88/75	13	49	62
Note4:	21	26/21	5	11	16
Note5:	36	47/36	11	14	25
Note6:	61	84/61	23	15	38
Note7:	42	61/42	19	4	23
Note8:	47	72/47	25	3	22
Note9:	13	21/13	8	3	5
Note10:	17	29/17	12	7	5 (root 2 related)
Note 11:	89	160/89	71	53	18
Note 12	19	36/19	17	15	2
	29	55/29	26	23	3

	Base Note	Interval	Beats1*ζ1 (1/1)point	Beats2*ζ2 (3/2)point	Beats1*ζ3 (2/1)point
16EDO:					
Note1:	68	71/68	3	62	65
	113	118/113	5	103	108
Note2:	11	12/11	1	9	10
Note3:	29	33/29	4	21	25
	36	41/36	5	26	31
Note4:	16	19/16	3	10	13 (Fibonacci numbers)
	25	25/21	4	13	17
	37	44/37	7	23	30
Note5:	25	31/25	6	13	19
	29	36/29	7	15	22
Note6:	17	22/17	5	7	12 (root2 related)
	27	35/27	8	11	19
	37	48/37	11	15	26
	64	83/64	19	26	45
Note7:	17	23/17	6	5	11
	31	42/31	11	9	20
	48	65/48	17	14	31

Note8: squareroot(2)

Note9:	21	31/21	10	1	11
	44	65/44	21	2	23
	65	96/65	31	3	34
Note10:	11	17/11	6	1	5
	24	37/24	13	2	11
	59	91/59	32	5	27
Note11:	18	29/18	11	4	7 (Lucas numbers)
	59	95/59	36	13	23
Note12:	22	37/22	15	8	7
Note13:	33	58/33	25	17	8
	37	65/37	28	19	9
	41	72/41	31	21	10
Note14:	6	11/6	5	4	1
Note15:	12	23/12	11	10	1
	35	67/35	32	29	3
	47	90/47	43	39	4

	Base Note	Interval	Beats1*ζ1 (1/1)point	Beats2*ζ2 (3/2)point	Beats1*ζ3 (2/1)point
17EDO:					
Note1:	24	25/24	1	22	23
Note2:	47	51/47	4	39	43
Note3:	23	26/23	3	17	20 (Fibonacci Hybrid numbers)
Note4:	17	20/17	3	11	14
	45	53/45	8	29	37
Note5:	22	27/22	5	12	17
Note6:	18	23/18	5	8	13 (Fibonacci numbers)
	47	60/47	13	21	34
	65	84/65	18	29	47
Note7:	3	4/3	1	1	2
Note8:	13	18/13	5	3	8
	44	61/44	17	10	27
	57	79/57	22	13	35
	70	97/70	27	16	43
Note 9:	9	13/9	4	1	5
Note10:	147	221/147	74	1	73
Note11:	23	36/23	13	3	10
	30	47/30	17	4	13 (Fibonacci Hybrid numbers)
	53	83/53	30	7	23
Note12:	19	31/19	12	5	7
Note 13:	10	17/10	7	4	3 (Lucas Hybrid numbers)
	53	90/53	37	21	16
Note14:	13	23/13	10	7	3
Note15:	13	24/13	11	9	2
	19	35/19	16	13	3 (Fibonacci Hybrid Numbers)
Note 16:	13	25/13	12	11	1
	25	48/25	23	21	2

	Base Note	Interval	Beats1*ζ1 (1/1)point	Beats2*ζ2 (3/2)point	Beats1*ζ3 (2/1)point
19EDO:					
Note1:	27	28/27	1	25	26
Note2:	40	43/40	3	34	37
Note3:	17	19/17	2	13	15
Note4:	13	15/13	2	9	11 (Lucas Hybrid Numbers)
	19	22/19	3	13	16 (Fibonacci Hybrid numbers)
Note5:	5	6/5	1	3	4
Note6:	4	5/4	1	2	3
	29	36/29	7	15	22
	37	46/37	9	10	28
	45	56/45	11	23	34
Note7:	17	22/17	5	7	12 (Root2 numbers)
	24	31/24	7	10	17
	31	40/31	9	13	22
	55	71/55	16	23	39
Note8:					
	50	67/50	17	16	33
	121	162/121	41	39	80
	180	241/180	61	58	119
	239	320/239	81	77	158
Note9:	18	25/18	7	4	11 (Lucas Numbers)

Note10:	16	23/16	7	2	9
	25	36/25	11	3	14
Note11:	81	121/81	40	1	41
	160	239/160	79	2	81
Note12:	11	17/11	6	1	5
	20	31/20	11	2	9
	51	79/51	28	5	23
Note13:	5	8/5	3	1	2
	18	29/18	11	4	7 (Lucas Numbers)
	23	37/23	14	5	9
	28	45/28	17	6	11
Note14:	3	5/3	2	1	1
Note15:	11	19/11	8	5	3 (Fibonacci Numbers)
Note16:	5	9/5	4	3	1
	14	25/14	11	8	3
	19	34/19	15	11	4
	24	43/24	19	14	5
	29	52/29	23	17	6
Note17:	7	13/7	6	5	1
	36	67/36	31	26	5
	50	93/50	43	36	7
	64	119/64	55	46	9
Note 18	14	27/14	13	12	1

	Base Note	Interval	Beats1*ζ_1 (1/1)point	Beats2*ζ_2 (3/2)point	Beats1*ζ_3 (2/1)point
22EDO:					
Note1:	94	97/94	3	88	91
	125	129/125	4	117	121
Note2:	31	33/31	2	27	29
	46	49/46	3	40	43
Note3:	10	11/10	1	8	9
	81	89/81	8	65	73
Note4:	15	17/15	2	11	13
	37	42/37	5	27	32
	52	59/52	7	38	45
	67	76/67	9	49	58
Note5:	6	7/6	2	8	10
	23	27/23	4	15	19
	29	34/29	5	19	24
	35	41/35	6	23	29
	41	48/41	7	27	34
Note6:	19	23/19	4	11	15
	24	29/24	5	14	19
Note7:	4	5/4	1	2	3
	! 77	96/77	19	39	58
Note8:	7	9/7	2	3	5
Note9:	3	4/3	1	1	2
	61	81/61	20	21	41
Note10:	8	11/8	3	2	5
	19	26/19	7	5	12 (Root2 numbers)
	! 27	37/27	10	7	17 (Root2 Numbers)

Note11: Squareroot(2)

Note12:	11	16/11	5	1	6
	13	19/13	6	1	7
	24	35/24	11	2	13
	37	54/37	17	3	20
Note13:	2	3/2	1	0	1
	81	122/81	41	1	40
Note14:	9	14/9	5	1	4
Note15:	5	8/5	3	1	2
	48	77/48	29	10	19
Note 16:	29	48/29	19	9	10
	61	101/61	40	19	21
Note17:	7	12/7	5	3	2
	17	29/17	12	7	5 (root2 Numbers)
	24	41/24	17	10	7 (root2 Numbers)
Note18:	17	30/17	13	9	4
	21	37/21	16	11	5
	38	67/38	39	20	9
Note 19:	11	20/11	9	7	2
	39	71/39	32	25	7
Note20:	8	15/8	7	6	1
	25	47/25	22	19	3
	33	62/33	29	25	4
Note 21:	16	31/16	15	14	1
	81	157/81	76	71	5

Some Sharp & Flat perfect fifths for EDOs 3 to 31 with Beating Ratios

So using an algorithm to find closer approximations to a mistuned (3/2)=perfect fifth interval for some EDOs (Equal divisions per Ocatve),
we found before for an interval = I
Beat frequencies associated with most prominent points in theory (1/1) & (3/2) and (2/1) are:

$a\zeta_1 = a.I\text{-}b$ beat frequency associated with point (b/a)

$g\zeta_2 = h\text{-}gI$ beat frequency associated with point (h/g) for the 'far left region'
and for this term we know that for the far right region $g\zeta_2 = gI\text{-}h$, simply a + or - value

$p\zeta_3 = q\text{-}pI$

So if we let (b/a) = 1/1, (h/g) = 3/2, (q/p) = 2/1
For the next data:

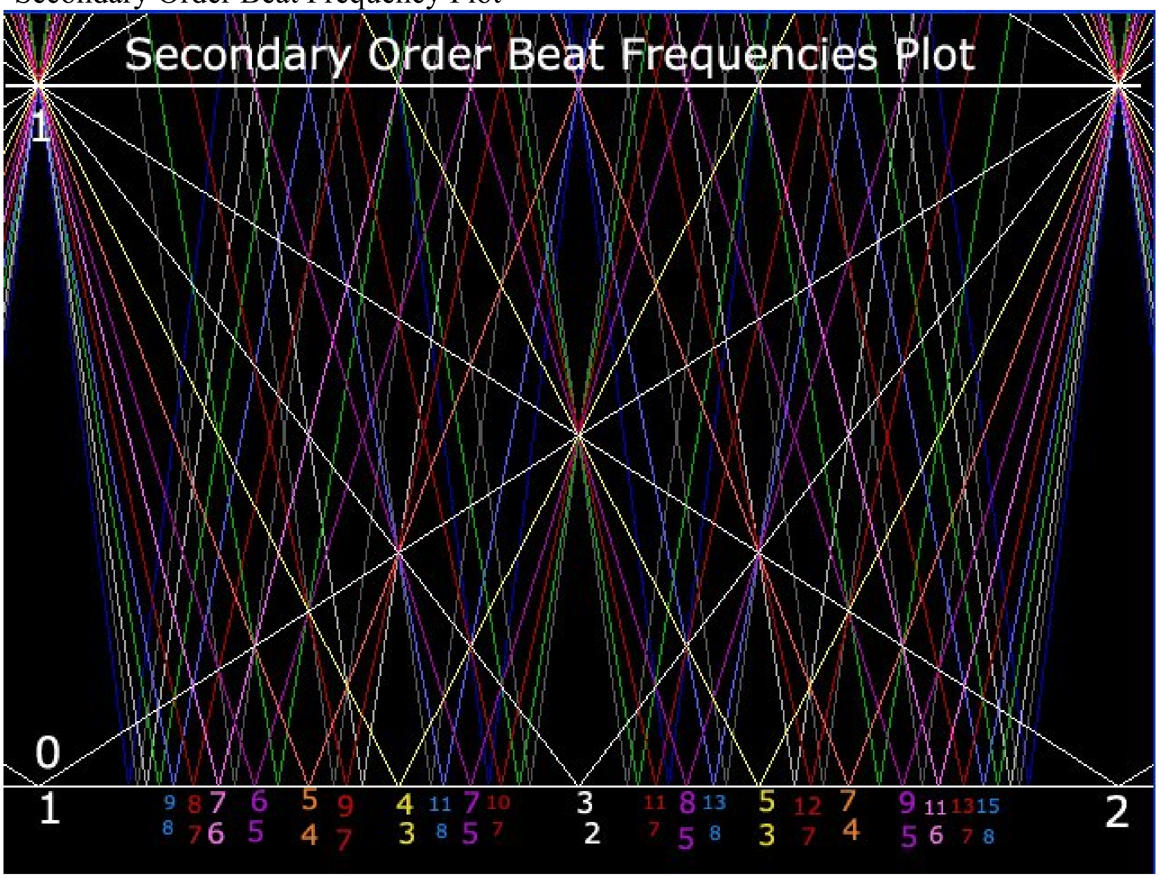
Interval	Beats associated with (1/1)	Beats associated with(3/2)	Beats associated with (2/1)
31EDO			
flat fifth 2^(18/31) = 335/224	111	2	113
29EDO:			
sharp fifth 2^(17/29) = 581/387	194	1	193
27EDO			
sharp fifth 2(16/27) = 95/63	32	1	31
26EDO			
flat fifth 2^(15/26) = 91/61	30	1	31
179/120	59	2	61
25EDO			
sharp fifth 2^(15/25)= 97/64	33	2	31
144/95	49	3	46
23EDO			
sharp fifth 2^(14/23) =32/21	11	1	10
61/40	21	2	19
flat fifth 2^(14/23) = 37/25	12	1	13
108/73	35	3	38
145/98	47	4	51
22EDO			
sharp fifth 2^(13/22) = 122/81	41	1	40
485/322	163	4	159
851/565	286	7	279
21EDO			
flat fifth 2^(12/21) = 52/35	17	1	18
107/72	35	2	37
20EDO			
flat fifth 2^(11/20) = 41/28	13	2	15

	Interval	Beats associated with (1/1)	Beats associated with(3/2)	Beats associated with (2/1)
19EDO				
flat fifth 2^(11/19) =	121/81	40	1	41
	239/160	79	2	81
	838/561	277	7	284
17EDO				
sharp fifth 2^(10/17) =	221/147	74	1	73
16EDO				
sharp fifth 2^(10/16) =	17/11	6	1	5
	37/24	13	2	11
	91/59	32	5	27
	128/83	45	7	38
flat fifth 2^(9/16) =	31/21	10	1	11
	65/44	21	2	23
	96/65	31	3	34
15EDO				
sharp fifth 2^(9/15) =	97/64	33	2	31
	144/95	49	3	46
13EDO				
flat fifth 2^(7/13) =	61/42	19	4	23
sharp fifth 2^(8/13)=	49/32	17	2	15
	72/47	25	3	22
And for 12ET:				
sharp fifth 2^(7/12)=	442/295	147	1	148
next order is:	1329/887	442	3	445
7ET				
flat fifth 2^(4/7) =	52/35	17	1	18
	107/72	35	2	37
	159/107	52	3	55
	584/393	191	11	202
5ET				
sharp fifth 2^(3/5)				
Series =	26/17	9	1	8
	29/19	10	1	9
	32/21	11	1	10
	35/23	12	1	11
	38/25	13	1	12
	41/27	14	1	13
	44/29	15	1	14
	47/31	16	1	15
	97/64	33	2	31

Example: Using 22EDO

Near perfect fifth interval = 2^(13/22) approximately = 122/81

So we may have a hypothesis that a secondary order beating frequency (or called additional wave structure periodicity) associated with, connected with a just intonation ratio point (b/a) is present for any arbitrary interval, so an infinite number of beat frequencies co-exist and are perceived together at a variety of relative intensities, for all values of reduced integer fraction just intoned ratio (b/a).
This is at the moment conlcluded by audible perception of sound experiments generating sine-waves for certain intervals and observing the wave structural periodicity visually and listening audibly. So (b/a) seem to need to be in reduced form as shown in the picture : 'Secondary Order Beat Frequency Plot'



So we have an approximation of the close perfect fifth in 22EDO ,2^(13/22) = 122/81

We made a table before, showing the Secondary Order Beat Frequency integers associated with the just (1/1) point, the just (3/2) frequency point and the just (2/1) frequency point. Let's see what integers crop up for other not so prominently perceived beat frequency associated points. Ie For (b/a), a the gradient is a larger integer. And remembering (b/a) is a 'reduced' integer fraction.

So what about the beats produced associated with the (4/3) point and (5/3) point?
Well 2^(13/22) = (4/3)+ζ and associated beating freq =3*ζ
Related integer fitting in the table = 81*(3*ζ) = 42.0055 ~ 42!
And with (5/3)+ζ [different ζ] associated beating freq integer =81*(3*ζ)=
=38.994...~39 !

Wow let's push our luck and test further beating frequencies associated with (5/4) & (7/4)
2^(13/22) = (5/4)+ζ and associated beating freq =4*ζ (again new ζ)
Related integer fitting in the table = 81*(4*ζ) = 83.007 ~ 83!

And with (7/4) ,
2^(13/22) = (7/4)+ζ with another related new ζ, Related integer fitting in the table =
= 81*(4*ζ) = 78.993... ~ 79!

Looking good, let's try next associated beating points (6/5), (7/5), (8/5), (9/5)
2^(13/22) = (6/5)+ζ and associated beating freq =5*ζ*81 (another new ζ)
Related beating integer fitting into table =124.009 ~124!

2^(13/22) = (7/5)+ζ and associated beating freq =5*ζ (another new ζ)
Related beating integer fitting into table =43.009 ~43!

2^(13/22) = (8/5)+ζ and associated beating freq =5*ζ (another new ζ)
Related beating integer fitting into table =37.991 ~38!

2^(13/22) = (9/5)+ζ and associated beating freq =5*ζ (another new ζ)
Related beating integer fitting into table =118.991 ~119!

2^(13/22) = (7/6)+ζ and associated beating freq =6*ζ (New ζ)
Related beating integer fitting into table =165.011*81 ~165!

2^(13/22) = (11/6)+ζ and associated beating freq =6*ζ*81 (New ζ)
Related beating integer fitting into table =158.989 ~159!

Surely not with (8/7),(9/7),(10/7),(11/7),12/7),(13/7) associated beating points too?
2^(13/22) = (8/7)+ζ and associated beating freq =7*ζ*81 (New ζ)
Related beating integer fitting into table =206.01288 ~206!

2^(13/22) = (9/7)+ζ and associated beating freq =7*ζ*81 (New ζ)
Related beating integer fitting into table =125.013 ~125!!

2^(13/22) = (10/7)+ζ and associated beating freq =7*ζ*81 (New ζ)
Related beating integer fitting into table =44.013 ~44!

2^(13/22) = (11/7)+ζ and associated beating freq =7*ζ*81 (New ζ)

Related beating integer fitting into table =44.013 ~44!

2^(13/22) = (12/7)+ζ and associated beating freq =7*ζ*81 (New ζ)
Related beating integer fitting into table =117.987 ~118!

2^(13/22) = (13/7)+ζ and associated beating freq =7*ζ*81 (New ζ)
Related beating integer fitting into table =198.987 ~199!

We'll try some more (9/8),(11/8),(13/8),(15/8)
2^(13/22) = (9/8)+ζ and associated beating freq =8*ζ*81 (New ζ)
Related beating integer fitting into table =247.015 ~247!

2^(13/22) = (11/8)+ζ and associated beating freq =8*ζ*81 (New ζ)
Related beating integer fitting into table =85.015 ~85!

2^(13/22) = (13/8)+ζ and associated beating freq =8*ζ*81 (New ζ)
Related beating integer fitting into table =76.985 ~77!

2^(13/22) = (15/8)+ζ and associated beating freq =8*ζ*81 (New ζ)
Related beating integer fitting into table =238.985 ~239!

And beats associated with points (10/9),(11/9),(13/9),(14/9),(16/9),(17/9)

2^(13/22) = (10/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =288.017 ~289

2^(13/22) = (11/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =207.016 ~207

2^(13/22) = (13/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =45.0165 ~45

2^(13/22) = (14/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =35.983 ~36

2^(13/22) = (16/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =197.983 ~198

2^(13/22) = (17/9)+ζ and associated beating freq =9*ζ*81 (New ζ)
Related beating integer fitting into table =278.983 ~279

And next for beats associated with points (11/10),(13/10),(17/10),(19/10)
2^(13/22) = (17/10)+ζ and associated beating freq =10*ζ*81 (New ζ)
Related beating integer fitting into table = 156.9816~157
So will all associated beat integer fraction points produce integers?

Note the common occurrence of the decimal value for a fixed value of bottom integer, ‘a’ and variable top integer, ‘b’ for reduced just ratio (b/a) the associated beat frequency point. Will be talked about further on…

We want a result of a top part of fraction as an integer, and a bottom part of fraction as an integer.
Interval I, like we said can be expressed, I = (b/a)+ζ where b & a are integers in a reduced fraction (b/a) „,& ζ is a small change although really can be any size, but we are really dealing with mistuned fifths, octaves, thirds, sixths , perfect, minor , major etc..

So (b/a)+ζ = [Iup]/[Idown]
, Iup being top part of fraction for interval value , I, & Idown being bottom part .

rearranging a*ζ = (a*[Iup]/[Idown])-b

= (a*[Iup]-b*[Idown]) / [Idown]

Very simple stuff

Using the 22EDO example 2^(13/22) ~ 122/81

What is the lower note maybe called the fundamental = frequency integer = 1 could be assigned = 81
So all is multiplied by 81 and 81 is the [Idown] bottom part in interval as integer fraction form.

So beat frequency integer (which could be called additional wave structure periodicity frequency as before), aζ=
The simple result is:
Beat frequency integer = a*ζ = a*[Iup]-b*[Idown]

remembering (b/a) is the reduced Just ratio associated with a component Beat frequency

So in 22EDO mistuned ‘perfect fifth’ = Interval, I = 2^(13/22)~122/81, [Iup] = 122, [Idown] = 81
as said Beat frequency integer = a*ζ = a*[Iup]-b*[Idown]
So if 1 = a*122-b*81
(b/a) = 3/2

Then if 2 = a*122-b*81
(b/a) = ?/?

This needs a bit more work...

Beating Symmetric Triads

Some nice triad chords in balance appear here:
Introducing further beating frequencies or same as additional wave structure periodicity, (4/3) & (5/3) i.e. the next increment of the bottom integer in the just ratio.
So like before:
interval, I = (b/a)+ζ
Beats with (b/a) = a*ζ =a*I-b

Beats with (1/1) =1*I-1
Beats with (4/3) =3*I-4
Beats with (3/2) =2*I-3
Beats with (5/3) =3*I-5
Beats with (2/1) =1*I-2

Interval,I	Beats With (1/1)	Beats with (4/3)	Beats with (3/2)	Beats with (5/3)	Beats with (2/1)

7/6…..........1/6..........3/6............4/6...........9/6...........5/6
Reflection interval about symmetry point (3/2) so that the values become reflected, swapped
11/6............5/6..........9/6............4/6...........3/6...........1/6
(11/6)/(7/6) = 11/7
So continuing table to find the beats generated with the 11/7 interval when a triad is played:
11/7............4/7..........5/7............1/7............2/7...........3/7
We need to multiply the beats results for (11/7) by (7/6) to find the true beat frequencies:
(11/7)part*(7/6)
...................4/6..........5/6.............1/6............2/6..........3/6
Try another:

Interval,I	Beats With (1/1)	Beats with (4/3)	Beats with (3/2)	Beats with (5/3)	Beats with (2/1)

8/7.............1/7...........4/7............5/7..........11/7...........6/7
Reflection interval about (3/2) point
13/7...........6/7..........11/7...........5/7...........4/7............1/7
other interval between (13/7)/(8/7) = 13/8
So continuing table to find the beats generated with the 13/8 interval when a triad is played:
13/8………5/8………7/8……….2/8………1/8……….3/8
(13/8)part beats*(8/7)=
..................5/7...........1/1............2/7...........1/7............3/7

There appears to be a beating frequency symmetry about the point (3/2)

So for an interval , (b/a) sounded with 1 which lies before (3/2) then other mirror interval on the other side of (3/2)
=(3/2)-(b/a)+(3/2) = 3-(b/a)

So notes in the triad are
(a/a) : (b/a) : 3-(b/a)

multiplying by ‘a’ gives triad as:

a : b : (3a-b)

A symmetric triad about point (3/2)

This needs to be continued and investigated further…

We have checked the results using a computer algorithm and everything seems to be working and in check…

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Paul Erlich: Forms Of Tonality This paper briefly illustrates the concepts of tone-lattices, scales, periodicity and notational systems for 5-limit and 7-limit music. These concepts will be explored in much greater detail in a future work.**2001**

This work only exists because of 4 souls I wish to thank from the school of learning. A few errors were made and amended and still open to debate.. It seems one can never be completely certain about a theory, especially when comparing concepts of planetary motion with circular oscillations of sound.